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The initial-value problem for long waves of finite amplitude

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Derived herein is a set of partial differential equations governing the propagation of an arbitrary, long-wave disturbance of small, but finite amplitude. The equations reduce to that of Boussinesq (1872) when the assumption is made that the disturbance is propagating in one direction only. The equations are hyperbolic with characteristic curves of constant slope. The initial-value problem can be solved very readily by numerical integration along characteristics. A few examples are included.

1. Introduction

A difficult and important part of the theory of water waves concerns the propagation of disturbances of small or moderate height and of long wavelength. These two restrictions may be expressed mathematically by the inequalities

$$\frac{a}{h} \ll 1, \quad \frac{\lambda}{h} \gg 1, \tag{1}$$

where h is the uniform depth of the water, a is a length representative of the amplitude, and λ is a length representative of the wavelength of the disturbance.

Three different theories have been advanced to predict the development of wave disturbances of this kind. One by Airy (1845) makes the additional assumption that the pressure distribution is hydrostatic. It can be shown that Airy's theory does not permit long waves of finite amplitude to propagate without change of shape. The disturbances eventually steepen and break, no matter how small they are to begin with. It may be shown that Airy's theory implies the additional restriction that

$$\frac{a}{h}\frac{\lambda^2}{h^2} \gg 1. \tag{2}$$

Another theory, by Jeffreys & Jeffreys (1946), is valid when

$$\frac{a}{h}\frac{\lambda^2}{h^2} \ll 1. \tag{3}$$

Finally, however, there is an equation, given first by Boussinesq, which governs the propagation of long waves when

$$\frac{a}{h}\frac{\lambda^2}{h^2} \sim 1. \tag{4}$$

It is
$$\frac{\partial^2\eta'}{\partial t'^2}-gh\frac{\partial^2\eta'}{\partial x'^2}=gh\left[\frac{3}{h}\left(\frac{\partial\eta'}{\partial x'}\right)^2+\frac{3\eta'}{h}\frac{\partial^2\eta'}{\partial x'^2}+\frac{h^2}{3}\frac{\partial^4\eta'}{\partial x'^4}\right], \tag{5}$$
where η' is the height of the surface above the undisturbed level, g is gravity, x' is horizontal distance, and t' is time. The Boussinesq theory includes the theories of Airy and Jeffreys & Jeffreys as special cases.

The above interpretation of the long-wave problem was first made by Ursell (1953; see also Benjamin & Lighthill 1954), who gave an independent derivation of Boussinesq's equation. As Ursell, and Keulegan & Patterson (1940) have pointed out, the Boussinesq equation is satisfied by the solitary wave (Rayleigh 1876), which is precisely a long wave which *does* propagate without change of shape. Consequently, Ursell's paper, in pointing out the importance of the quantity $a\lambda^2/h^3$, finally explained the *long-wave paradox* which consisted of the inapplicability of the Airy theory to the solitary wave.

An important restriction which was imposed quite obviously in Keulegan & Patterson's development of Boussinesq's equation, and not so obviously in Ursell's development, is that the Boussinesq equation does not govern the propagation of an arbitrary long-wave disturbance, but, rather, holds only when the wave heights propagate in one direction only, at a speed nearly equal to $(gh)^{\frac{1}{2}}$. Korteweg & de Vries (1895) also considered the initial-value problem for long waves, but again the analysis was restricted to disturbances moving in one direction only. The purpose of this paper is to derive a set of equations which predicts the development of an arbitrary, small-amplitude, long-wave disturbance. It will be shown that the present theory reduces to that of Boussinesq when the restriction is made that the disturbances propagate in one direction only.†

2. The differential equations of long gravity waves

Figure 1 represents a portion of a long-wave disturbance.‡ The motion is two-dimensional, the undisturbed height of the water is h, and η' is the height of the disturbance above the undisturbed level. We assume that the motion is irrotational with a velocity potential ϕ', which satisfies
$$\nabla^2\phi' = 0. \tag{6}$$
At the surface the pressure is constant, and the dynamic equation is
$$g\eta'+\tfrac{1}{2}(u_s'^2+v_s'^2)-\left(\frac{\partial\phi'}{\partial t'}\right)_s = 0, \tag{7}$$
where we have assumed that there is some section at which
$$\eta' = u_s' = v_s' = (\partial\phi'/\partial t')_s = 0.$$

† Meyer (1962) discusses the interaction of two solitary waves moving through each other from opposite directions. He uses a differential equation which really applies to disturbances propagating in one direction only, but Meyer's result that the interaction is linear still follows from a slightly altered argument.

‡ In this paper dimensional variables are denoted by primed symbols, non-dimensional quantities by unprimed symbols.

An important special case is when the fluid is at rest and remains at rest in some part of the region.

The equation of the surface is

$$y' - h - \eta'(x', t') = 0, \tag{8}$$

so that the kinematic condition at the surface is

$$v'_s = \frac{\partial \eta'}{\partial t'} + u'_s \frac{\partial \eta'}{\partial x'}. \tag{9}$$

In addition, at the bottom, $y' = 0$, the vertical component of the velocity must vanish.

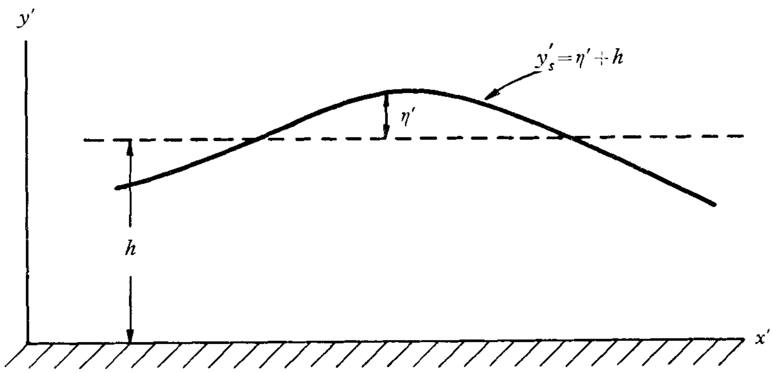

FIGURE 1. Long-wave disturbance.

In his derivation of the solitary wave, Rayleigh assumed that the complex potential for the motion is given by the

$$\phi' + i\psi' = F' + (iy')F'_{x'} + \frac{(iy)^2}{2!} F'_{x'x'} + \frac{(iy)^3}{3!} F'_{x'x'x'} + \ldots, \tag{10}$$

where the right-hand side of equation (10) represents the Taylor series expansion of a function of a complex variable $x' + iy'$ about $y' = 0$. In addition to satisfying the kinematic condition at the bottom of the channel, this expansion is inherently suitable for the complex potential of a long wave, because the changes in such a wave are very gradual in the x'-direction, and the higher derivatives with respect to x' are small. In Rayleigh's case, the motion is stationary with respect to a co-ordinate system moving with the wave, so that the function F' is a function of x' only. Here, however, the disturbances are unsteady, and we permit F' to be a function of time as well.

The real part of equation (10) is

$$\phi'(x', y', t') = F'(x', t') - (y'^2/2!) F'_{x'x'}(x', t') + \ldots \tag{11}$$

We now make all quantities dimensionless as follows:

$$\left.\begin{aligned} x = x'/h, \quad y = y'/h, \quad t = t'\surd(g/h), \\ \phi(x,y,t) = \phi'(x',y',t')/h\surd(gh), \quad \eta = \eta'/h, \\ F(x,t) = F'(x',t')/h\surd(gh). \end{aligned}\right\} \tag{12}$$

Our two fundamental equations (7) and (9) can now be written as

$$\eta = (\phi_t)_s - \tfrac{1}{2}(\phi_x^2 + \phi_y^2)_s, \tag{13}$$

$$\eta_t = -(\phi_y)_s + \eta_x(\phi_x)_s. \tag{14}$$

In an earlier paper (Long 1956) it was shown that a fundamental assumption in the derivation of the solitary wave was that, if the non-dimensional amplitude of the disturbance is of the order of α, i.e. if

$$\eta \sim \alpha, \tag{15}$$

then derivatives in the x-direction have the order of magnitude†

$$\partial/\partial x \sim \alpha^{\frac{1}{2}}. \tag{16}$$

The same assumption is made here for the unsteady case. Obviously (16) is in accordance with the fundamental property of long waves as expressed in (4). An additional assumption, also carried over from the steady-state problem, is that

$$U \equiv -F_x \sim \alpha. \tag{17}$$

Notice that U is the velocity at the bottom of the channel.

One additional assumption must be made here regarding the order of magnitude of the partial derivative with respect to t. In the case of the solitary wave, which has a non-dimensional speed of propagation

$$c = 1 + \tfrac{1}{2}\alpha, \tag{18}$$

the time derivative is related to the space derivative by the equation

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial x}. \tag{19}$$

In our case, disturbances propagate in both directions and the time and space derivatives are not simply related. Instead, we assume

$$\frac{\partial}{\partial t} \sim \frac{\partial}{\partial x}. \tag{20}$$

This is certainly satisfied when the disturbance is sufficiently small.

We now use the non-dimensional form of (11) to obtain from equations (13) and (14) two equations in two unknowns, η and F. Thus

$$\phi = F - \frac{y^2}{2!}F_{xx} + \frac{y^4}{4!}F_{xxxx} - \dots, \tag{21}$$

$$\phi_x = F_x - \frac{y^2}{2!}F_{xxx} + \frac{y^4}{4!}F_{xxxxx} - \dots, \tag{22}$$

$$\phi_y = -yF_{xx} + (y^3/3!)F_{xxxx} - \dots, \tag{23}$$

$$\phi_t = F_t - \frac{y^2}{2!}F_{xxt} + \frac{y^4}{4!}F_{xxxxt} - \dots. \tag{24}$$

† Here, the small quantity α will appear in the initial conditions. It must appear in such a way that the initial disturbance has the properties (15), (16) and (17).

Equations (13) and (14) now become

$$\eta = F_t - \frac{(1+\eta)^2}{2} F_{xxt} + \frac{(1+\eta)^4}{24} F_{xxxxt} - \dots$$
$$- \tfrac{1}{2}[F_x^2 - (1+\eta)^2 F_x F_{xxx} + (1+\eta)^2 F_{xx}^2 + \dots], \qquad (25)$$

$$\eta_t = (1+\eta) F_{xx} - \tfrac{1}{6}(1+\eta)^3 F_{xxxx} + \dots + \eta_x [F_x - \tfrac{1}{2}(1+\eta)^2 F_{xxx} + \dots]. \qquad (26)$$

If we neglect quantities of the order of α^2 in (25) and $\alpha^{\frac{5}{2}}$ in (26), we get

$$\eta = F_t + O(\alpha^2), \qquad (27)$$

$$\eta_t = F_{xx} + O(\alpha^{\frac{5}{2}}). \qquad (28)$$

To this order we have, therefore,

$$\eta_{tt} - \eta_{xx} = O(\alpha^3). \qquad (29)$$

This is the classical wave equation for the propagation of long, infinitesimal disturbances. If we retain one more order of magnitude in each of equations (25) and (26), we get†

$$\eta = F_t - \tfrac{1}{2}F_{xxt} - \tfrac{1}{2}F_x^2 + O(\alpha^3), \qquad (30)$$

$$\eta_t = F_{xx} + \eta F_{xx} - \tfrac{1}{6}F_{xxxx} + F_x \eta_x + O(\alpha^{\frac{7}{2}}). \qquad (31)$$

Notice that in equation (30) we may substitute for F_{xxt} the term η_{tt} and commit an error no greater than that already made in the derivation of our equations. Similar substitutions may be made in the other second-order terms, so that our differential equations may be written

$$\eta = F_t - \tfrac{1}{2}\eta_{tt} - \tfrac{1}{2}F_x^2 + O(\alpha^3), \qquad (32)$$

$$\eta_t = F_{xx} + \eta\eta_t - \tfrac{1}{6}\eta_{txx} + F_x \eta_x + O(\alpha^{\frac{7}{2}}). \qquad (33)$$

Finally, with the definitions

$$U = -F_x, \qquad (34)$$

$$\omega = \eta_t, \qquad (35)$$

$$\Omega = \eta_{tt}, \qquad (36)$$

we may obtain a set of four first-order equations

$$\eta_t + U\eta_x + U_x - \eta\eta_t + \tfrac{1}{6}\Omega_t = 0, \qquad (37)$$

$$\eta_x + U_t - U\eta_t + \tfrac{1}{2}\Omega_x = 0, \qquad (38)$$

$$\eta_t - \omega = 0, \qquad (39)$$

$$\omega_t - \Omega = 0. \qquad (40)$$

In the above, we have made the approximations

$$\eta_{xxt} = \eta_{ttt} + O(\alpha^{\frac{7}{2}}), \qquad (41)$$

$$UU_x = -U\eta_t + O(\alpha^{\frac{7}{2}}). \qquad (42)$$

† One may derive the results of this section by incorporating α into the non-dimensional definitions of (12) and then seeking a solution in the form of an expansion in the small parameter α. For example, the non-dimensional F is then $F'(x', t')/\alpha^{\frac{1}{2}}g^{\frac{1}{2}}h^{\frac{3}{2}}$ instead of $F'(x', t')/g^{\frac{1}{2}}h^{\frac{3}{2}}$. The differential equations of the first two approximations can then be combined to form the two equations (30) and (31). This approach makes unnecessary the tentative assumptions that $\eta \sim \alpha$, $\partial/\partial x \sim \alpha^{\frac{1}{2}}$, $U \sim \alpha$, $\partial/\partial t \sim \alpha^{\frac{1}{2}}$, but it has the disadvantage of being lengthy.

Equations (37)–(40) are the fundamental differential equations governing the propagation of long-wave disturbances. They reduce to equation (5), derived by Boussinesq, if we substitute

$$\frac{\partial}{\partial t} = \pm\frac{\partial}{\partial x} \tag{43}$$

in the second-order terms. Physically, this implies that the wave disturbances are all moving in one direction at a speed nearly equal to the speed of propagation of infinitesimal long waves.

In addition, we may show that equations (37)–(40) have a solution yielding the solitary wave. Thus, suppose that

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial x}. \tag{44}$$

Equations (37) and (38) then become

$$\eta_x = cU_x - \tfrac{1}{2}c^2\eta_{xxx} - UU_x, \tag{45}$$

$$-c\eta_x = -U_x - c\eta\eta_x + \tfrac{1}{6}c\eta_{xxx} - U\eta_x. \tag{46}$$

If we make the approximation

$$U\eta_x = cUU_x + O(\alpha^{\frac{7}{2}}), \tag{47}$$

these may be integrated to yield

$$\eta = cU - \tfrac{1}{2}c^2\eta_{xx} - \tfrac{1}{2}U^2, \tag{48}$$

$$-c\eta = -U - \tfrac{1}{2}c\eta^2 + \tfrac{1}{6}c\eta_{xx} - \tfrac{1}{2}cU^2, \tag{49}$$

where we have imposed the condition that the fluid is at rest at infinity. The linear theory shows that c is practically equal to 1. Therefore, if we let $c = 1 + b\alpha$, equations (48) and (49) become

$$\eta = U + b\alpha U - \tfrac{1}{2}\eta_{xx} - \tfrac{1}{2}\eta^2, \tag{50}$$

$$\eta = U - b\alpha\eta + \eta^2 - \tfrac{1}{6}\eta_{xx}. \tag{51}$$

Subtracting, we get

$$2b\alpha\eta - \tfrac{1}{3}\eta_{xx} - \tfrac{3}{2}\eta^2 = 0. \tag{52}$$

With the substitutions,

$$\eta_x = \delta, \quad \eta_{xx} = \frac{d}{d\eta}\left(\frac{\delta^2}{2}\right) \tag{53}$$

this may be written

$$-\frac{1}{6}\frac{d}{d\eta}(\delta^2) - \tfrac{3}{2}\eta^2 + 2b\alpha\eta = 0. \tag{54}$$

Equation (54) has the integral

$$-\tfrac{1}{6}\delta^2 - \tfrac{1}{2}\eta^3 + b\alpha\eta^2 = 0, \tag{55}$$

where we have again used the condition that the fluid is at rest at infinity. At the crest we have

$$\delta = 0, \quad \eta = \alpha. \tag{56}$$

Therefore, $b = \frac{1}{2}$. This is the classical result for the solitary wave. The equation

$$-\tfrac{1}{6}\eta_x^2 - \tfrac{1}{2}\eta^3 + \tfrac{1}{2}\alpha\eta^2 = 0 \tag{57}$$

may be integrated again to yield the profile of the wave. Notice that the solitary wave is the only steady motiòn in which the fluid is at rest at either end of the infinite channel. This was apparently first noticed by Benjamin & Lighthill (1954), although it was pointed out by both Boussinesq and Keulegan that solitary wave motion appears to be a highly favoured type of disturbance.

3. The initial-value problem

It is of great interest to consider the development of an arbitrary long-wave disturbance as time progresses. We may do this by investigating the characteristics of the hyperbolic equations (37)–(40). Using the method outlined by Courant & Friedrichs (1948), we find that the system has characteristic curves which are straight lines with the slopes

$$\frac{dx}{dt} = 0, \quad 0, \quad \sqrt{3}, \quad -\sqrt{3}. \tag{58}$$

Therefore, with the new independent variables

$$z = x+\sqrt{3}t, \quad \bar{z} = x-\sqrt{3}t \tag{59}$$

the equations may be written in characteristic form as follows:

$$\frac{\partial}{\partial z}[\eta+\sqrt{3}U+\tfrac{1}{2}\Omega]+\sqrt{3}\,U\frac{\partial\eta}{\partial z} = \tfrac{1}{2}\omega\left(\sqrt{3}\eta+2U-\frac{2}{\sqrt{3}}\right), \tag{60}$$

$$\frac{\partial}{\partial \bar{z}}[-\eta+\sqrt{3}\,U-\tfrac{1}{2}\Omega]+\sqrt{3}\,U\frac{\partial\eta}{\partial \bar{z}} = \tfrac{1}{2}\omega\left(\sqrt{3}\eta-2U-\frac{2}{\sqrt{3}}\right), \tag{61}$$

$$\eta_t = \omega, \tag{62}$$

$$\omega_t = \Omega. \tag{63}$$

These may be solved very easily by integrating numerically along the characteristic lines in the (x, t)-plane. The initial data at $t = 0$ are the elevation η and the velocity U as functions of distance along the x-axis. It then follows from the differential equations that, to the present order of accuracy,

$$\omega(0) = -U_x(0), \tag{64}$$

$$\Omega(0) = \eta_{xx}(0). \tag{65}$$

4. Numerical integrations

A few integrations were performed on an IBM 7090 electronic computer to illustrate the feasibility of a numerical solution of the initial-value problem. The integrations were along characteristic curves. In all cases, the initial velocity was zero, and the elevation was symmetrical about $x = 0$.

The first example is shown in figure 2. The initial disturbance had an amplitude of 0·50. The portion that moved to the right developed into a disturbance with a crest of amplitude about 0·20. After some time a weak trough developed and also moved towards the right. The amplitude of the crest varied considerably as time went on, as shown in figure 3. The period of the oscillation was a little over 4 non-dimensional time units, and the overall tendency was for a decrease of crest amplitude. The speed of the crest was very variable, averaging 0·78 when the amplitude was a minimum, and 1·33 when the amplitude was a maximum. The average speed, after the two waves were well separated, was 1·08. The average speed of the trough was 0·95, although it too was very variable. The trough amplitude varied with time, its period being about 25 % shorter than the period of oscillation of the crest amplitude.

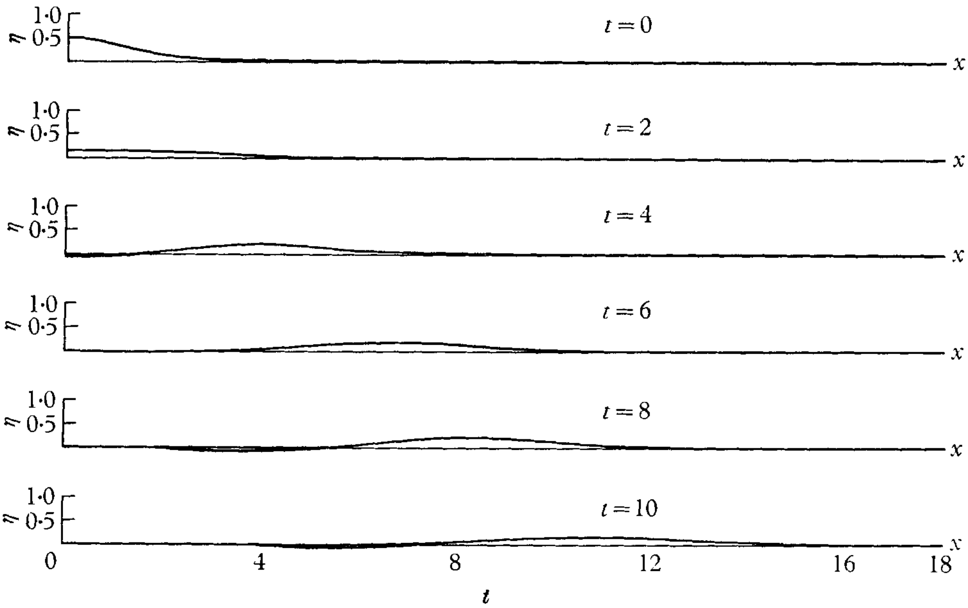

FIGURE 2. Profile of a wave disturbance of small amplitude. Distances are in units of $x = x'/h$, $y = y'/h$, and time in units of $t = t'\sqrt{(g/h)}$.

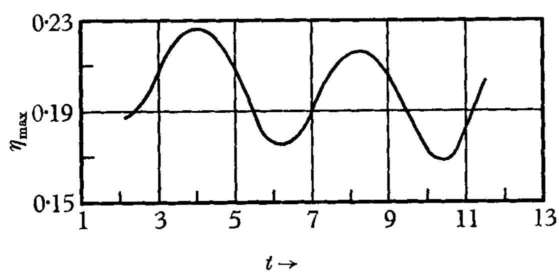

FIGURE 3. Amplitude of the crest as a function of time for the disturbance of figure 2.

On the whole the behaviour and properties of the main disturbance were very similar to those of the solitary wave except for the amplitude oscillation. As an example, in figure 4 the profile of a solitary wave of the same height, 0·217, is plotted on the same graph as the disturbance of $t = 8{\cdot}0$. The resemblance is quite good except that the computed disturbance is somewhat more peaked. In particular the wavelengths are about the same. The theoretical speed of the solitary wave is 1·10.

Figure 5 shows the development of a bigger initial disturbance. The main disturbance develops an amplitude of about 0·380. There was an indication of a periodic variation of amplitude, but the integration was too short to permit the kind of analysis shown in figure 3. The speed of the crest was about 1·38 when it

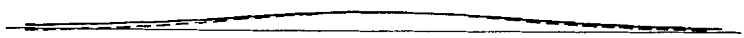

FIGURE 4. Profile of the disturbance of figure 2 at $t = 8$, and the solitary wave of the same amplitude.

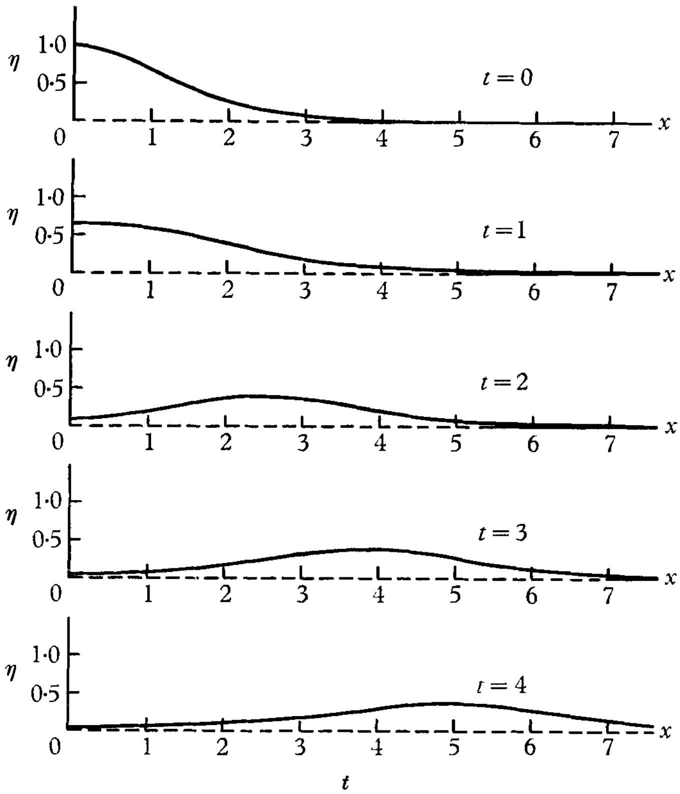

FIGURE 5. Profile of a wave disturbance of moderate amplitude. Distances are in units of $x = x'/h$, $y = y'/h$, and time in units of $t = t'\sqrt{(g/h)}$.

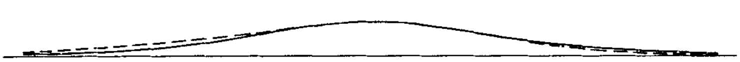

FIGURE 6. Profile of the disturbance of figure 5 at $t = 4$, and the solitary wave of the same amplitude.

first developed. It then slowed down to a speed of about 1·16 near the end of the integration. Finally, the main disturbance again resembled a solitary wave of the same amplitude. This is shown by the profile of a solitary wave of the same height, 0·39, plotted on the same graph as the disturbance of $t = 4$ (figure 6). The critical speed of this solitary wave is 1·18.

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